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Equilateral Convex Pentagons Which Tile the Plane

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It is shown that an equilateral convex pentagon tiles the plane if and only if it has two angles adding to 180° or it is the unique equilateral convex pentagon with angles A, B, C, D, E satisfying $A + 2B = 360°$, $C + 2E = 360°$, $A + C + 2D = 360°$ ($A \cong 70.88°$, $B \cong 144.56°$, $C \cong 89.26°$, $D \cong 99.93°$, $E \cong 135.37°$). © 1985 Academic Press, Inc.

Although the area of mathematical tilings has been of interest for a long time there is still much to be discovered. We do not even know which convex polygons tile the plane. Furthermore, for those polygons which do tile, new tilings are being found. It is known that all triangles and quadrilaterals tile the plane and those convex hexagons which do tile the plane have been classified. It is also known that no convex n-gon with $n \geqslant 7$ tiles.

In this paper we consider the problem of finding all equilateral convex pentagons which tile the plane. The upshot of our study is the following:

THEOREM. *An equilateral convex pentagon tiles the plane if and only if it has two angles adding to* 180°, *or it is the unique equilateral convex pentagon* X *with angles* A, B, C, D, E *satisfying* $A + 2B = 360°$, $C + 2E = 360°$, $A + C + 2D = 360°$ ($A \cong 70.88°$, $B \cong 144.56°$, $C \cong 89.26°$, $D \cong 99.93°$, $E \cong 135.37°$).

Thus the list of equilateral convex pentagons which tile, to be found in Schattschneider's paper [2], is complete. We also note that, with this theorem, the only convex polygons whose ability to tile is still in question are the nonequilateral convex pentagons.

It should be remarked that in obtaining this result we make no assumptions regarding periodicity of any tiling. (Yet it is a fact that every equilateral convex pentagon which tiles does so in a periodic manner.)

Our method of proof is interesting if only for the fact that it works only for the problem at hand—it could not, for instance, handle the problems of

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finding all convex pentagons or all equilateral convex hexagons which tile. In various places in the proof computer calculations are used.

With no further ado, let us begin.

1. Initial Reduction

We can suppose that any tiling of an equilateral convex pentagon is edge-to-edge. For if not, it has one or more "fault-lines." It is easy to see that such fault-lines are necessarily parallel to one another, and that there are at most a countable infinity of them. So the tiling can be slipped along the fault-lines to become edge-to-edge (see Fig. 1.).

Thus we need concern ourselves only with the ways in which the angles match up at the vertices of the tiling, in other words, relations between the angles A, B, C, D, E of the pentagon of the form

$$m_A A + m_B B + m_C C + m_D D + m_E E = 360^\circ,$$

where $m_A, \ldots, m_E$ are nonnegative integers. First we show that the set of possible relations is *finite*. In an equilateral-convex pentagon, each angle is

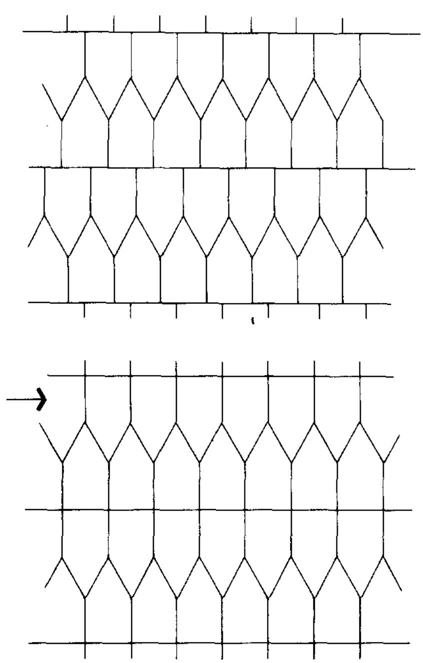


FIG. 1. Any tiling of an equilateral convex pentagon can be made edge-to-edge.

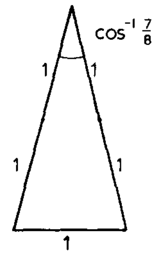


FIG. 2. In an equilateral convex pentagon, each angle is greater than $\cos^{-1}(\frac{7}{8})$.

greater than $\cos^{-1}(\frac{7}{8})$ (see Fig. 2), since if any angle were less than or equal to this, the polygon would fail to be a convex pentagon. So we have

$$A,\ B,\ C,\ D,\ E > \cos^{-1}(\tfrac{7}{8}) > 28^\circ$$

and since $m_A A + m_B B + \cdots + m_E E = 360^\circ$, we have

$$m_A + m_B + \cdots + m_E \leqslant 12.$$

Further, since A, B, C, D, E are all less than 180°,

$$m_A + \cdots + m_E \geqslant 3.$$

Thus there are a finite (if large) set of relations that might be satisfied by some equilateral convex pentagon. We proceed to show how the above list may be whittled down.

There is a lot of duplication; if, for example, a pentagon satisfies $A + 2B = 360^\circ$, it also satisfies, after suitable relettering, any one of ten different relations (assuming, as we shall from now on, that the angles of the pentagon are A, B, C, D, E in that order around the pentagon.) We remove such trivial duplication by adopting, without loss of generality, the following conventions. We shall suppose, in everything that follows, that B is the largest angle of the pentagon, that is, $B \geqslant A, C, D, E$, and, further, that of the two angles adjacent to B, namely A and C, A is the smaller, that is, $A \leqslant C$.

With the above conventions, we have

LEMMA 1. $A \leqslant C \leqslant D \leqslant E \leqslant B$.

Proof. Suppose B is fixed, and A decreases from its value when equal to C (see Fig. 3(a)) when D is equal to E. As A decreases, E increases, D decreases and C increases, until E becomes equal to B and C becomes equal to D (see Fig. 3(b)). Thus we have

$$A \leqslant C \leqslant \lim C = \lim D \leqslant D \leqslant E \leqslant B,$$

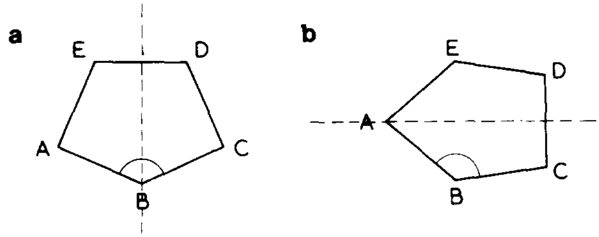


FIG. 3. (a) The greatest values of A occurs when $A=C$, $D=E$; (b) the least value of A occurs when $B=E$, $C=D$.

or

$$A \leqslant C \leqslant D \leqslant E \leqslant B. \quad \blacksquare$$

Lemma 1 will prove useful later in reducing the list of relations.

2. The Geometry of an Equilateral Convex Pentagon

In order to proceed we need to make a careful study of the geometry of the equilateral convex pentagon. Indeed, we prove

LEMMA 2. $108^\circ \leqslant B < 180^\circ$,

$$180^\circ - \tfrac{1}{2}B - \sin^{-1}(\sin(\tfrac{1}{2}B) - \tfrac{1}{2}) \geqslant A \geqslant 180^\circ - B + 2\sin^{-1}(1/4\sin(\tfrac{1}{2}B))$$

$$D = \cos^{-1}(\cos A + \cos B - \cos(A+B) - \tfrac{1}{2})$$

$$C = 270^\circ - B - \tfrac{1}{2}D + \theta$$

$$E = 270^\circ - A - \tfrac{1}{2}D - \theta,$$

where

$$\theta = \tan^{-1}((\sin A - \sin B)/(1 - \cos A - \cos B))$$

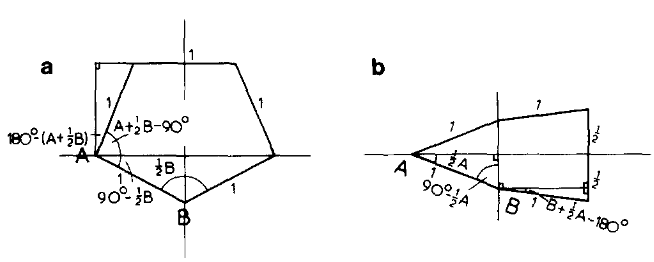


FIG. 4. (a) The greatest value of A is given by $\sin(\tfrac{1}{2}B) - \sin(180^\circ - (A + \tfrac{1}{2}B)) = \tfrac{1}{2}$; (b) the smallest value of A is given by $\sin(\tfrac{1}{2}A) + \sin(B + \tfrac{1}{2}A - 180^\circ) = \tfrac{1}{2}$.

Proof. The greatest value of A occurs when $A=C$ (see Fig. 4(a)), and then

$$\sin(\tfrac{1}{2}B)-\sin(180^\circ-(A+\tfrac{1}{2}B))=\tfrac{1}{2}$$

or

$$A=180^\circ-\tfrac{1}{2}B-\sin^{-1}(\sin(\tfrac{1}{2}B)-\tfrac{1}{2}).$$

The smallest value of A occurs when $B=E$ (see Fig. 4(b)), and then

$$\sin(\tfrac{1}{2}A)+\sin(B+\tfrac{1}{2}A-180^\circ)=\tfrac{1}{2}$$

or

$$2\sin(\tfrac{1}{2}A+\tfrac{1}{2}B-90^\circ)\cos(90^\circ-\tfrac{1}{2}B)=\tfrac{1}{2}$$

or

$$2\sin(\tfrac{1}{2}A+\tfrac{1}{2}B-90^\circ)\sin(\tfrac{1}{2}B)=\tfrac{1}{2},$$
$$A=180^\circ-B+2\sin^{-1}(1/4\sin(\tfrac{1}{2}B)).$$

In order to find D in terms of A and B, we calculate the length of the diagonal CE in two different ways. From Fig. 5(a) we see that

$$CE^2=(2\sin\tfrac{1}{2}D)^2$$

while from Fig. 5(*b*) we have

$$CE^2=(1-\cos A-\cos B)^2+(\sin A-\sin B)^2.$$

Equating these expressions yields

$$\cos D=\cos A+\cos B-\cos(A+B)-\tfrac{1}{2}.$$

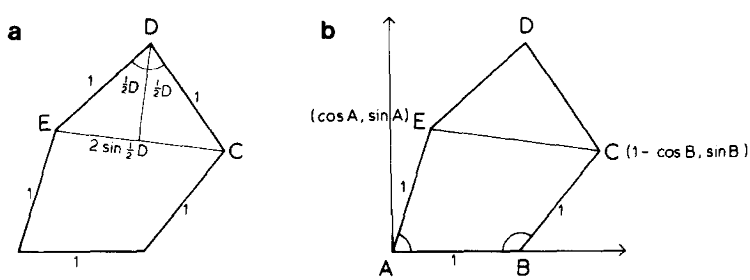


FIG. 5. (a) In an arbitrary equilateral convex pentagon the length of the diagonal CE is given by $CE=2\sin\frac{1}{2}D$; (b) an alternative expression for the length of the diagonal CE is given by $(CE)^2=(1-\cos A-\cos B)^2+(\sin A-\sin B)^2$.

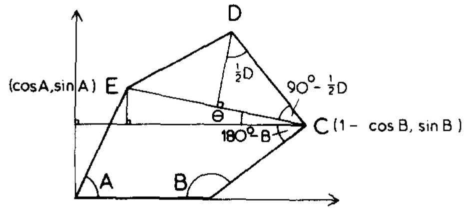


FIG. 6. This shows that $C = 270° - B - \frac{1}{2}D + \theta$, and hence that $E = 270° - A - \frac{1}{2}D - \theta$, where $\tan \theta = (\sin A - \sin B)/(1 - \cos A - \cos B)$.

Now from Fig. 6. it is easy to see that

$$C = (180° - B) + (90° - \tfrac{1}{2}D) + \theta = 270° - B - \tfrac{1}{2}D + \theta$$

and

$$E = (180° - A) + (90° - \tfrac{1}{2}D) - \theta = 270° - A - \tfrac{1}{2}D - \theta,$$

where

$$\tan \theta = (\sin A - \sin B)/(1 - \cos A - \cos B). \quad \blacksquare$$

In particular it follows from Lemma 2 that in an equilateral convex pentagon the angles C, D, and E are uniquely determined by the angles A and B, so we can identify the equilateral convex pentagon with a point in the AB plane. The region $\mathscr{P}$ in the AB plane which results from this identification is shown in Fig. 7. We indicate certain polygons on the boundary

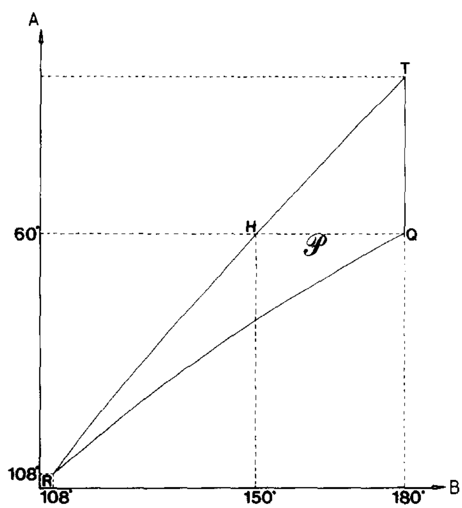


FIG. 7. Shows the region $\mathscr{P}$ in the AB plane which corresponds to the set of all equilateral convex pentagons. The part of the boundary in the line $B = 180°$ between T and Q consists of points representing quadrilaterals, not convex pentagons.

of the region. These are: R, the regular pentagon ($A=B=C=D=E=108°$); H, the house pentagon ($A=60°$, $B=E=150°$, $C=D=90°$); T, the isosceles triangle of Fig. 2 ($A=\cos^{-1}(\frac{7}{8})$, $B=E=180°$, $C=D=\cos^{-1}(\frac{1}{4})$); and Q, the quadrilateral which is half a regular hexagon ($A=C=60°$, $B=180°$, $D=E=150°$). Note that the part of the boundary of the region lying along the line $B=180°$ joining T and Q consists of (points representing) quadrilaterals, not convex pentagons.

3. Further Reduction

Now $\cos^{-1}(\frac{7}{8})((\cong 28.96°) < A \leqslant 108°$ and $108° \leqslant B < 180$ in $\mathscr{P}$. Also $60° < C \leqslant 108°$, $\cos^{-1}(\frac{1}{4})$ $(\cong 75.52°) < D < 120°$, and $108° \leqslant E < 180°$ for all points $(A, B) \in \mathscr{P}$. (Note that these constraints are determined by values at Q, R, and T). These constraints allow just 220 solutions to the equation

$$m_A A + m_B B + m_C C + m_D D + m_E E = 360°.$$

In order to determine which of our 220 relations are "good" in the sense that they are actually satisfied by some equilateral convex pentagon, we proceed as follows. For each set $(m_A, ..., m_E)$ we consider the function of A, B defined over the region $\mathscr{P}$ by $m_A A + \cdots + m_E E$, and which we will denote simply by $m_A m_B \cdots m_E$. (Thus, for example, 20010 denotes $2A + D$.) The relation $m_A A + \cdots + m_E E = 360°$ is good if and only if the function $m_A m_B \cdots m_E$ intersects the level set 360° for some $(A, B) \in \mathscr{P}$. However, rather than test each of the 220 functions in this way, we cut down the required work as follows: Define a partial order on the set of functions by writing

$$m_A \cdots m_E < m'_A \cdots m'_E$$

if $m_A A + \cdots + m_E E \leqslant m'_A A + \cdots + m'_E E$ simply by virtue of the fact that $A \leqslant C \leqslant D \leqslant E \leqslant B$. If $m_A m_B m_C m_D m_E < m'_A m'_B m'_C m'_D m'_E$ and $m'_A A + \cdots + m'_E E < 360°$ then $m_A A + \cdots + m_E E < 360°$. Hence we may discard such an $m_A \cdots m_E$. Similarly if $m_A m_B m_C m_D m_E > m'_A m'_B m'_C m'_D m'_E$ and $m'_A A + \cdots + m'_E E > 360°$ then we may discard $m_A m_B m_C m_D m_E$.

Lemma 3.

$$B + C + D < 360° \qquad A + C + D + E > 360°$$

$$2D + E < 360° \qquad 2A + 2C + E > 360°$$

$$2A + 2E > 360° \qquad 4C + D > 360°$$

$$A + B + 2C > 360° \qquad 4A + C + E > 360°$$

$$2A + 4C > 360^\circ$$
$$7A + E > 360^\circ$$
$$5A + 3C > 360^\circ$$
$$8A + 2C > 360^\circ$$
$$10A + C > 360^\circ$$

Proof. Figure 8 shows regions of the AB plane and demonstrates that the thirteen associated equalities do not meet $\mathscr{P}$. ∎

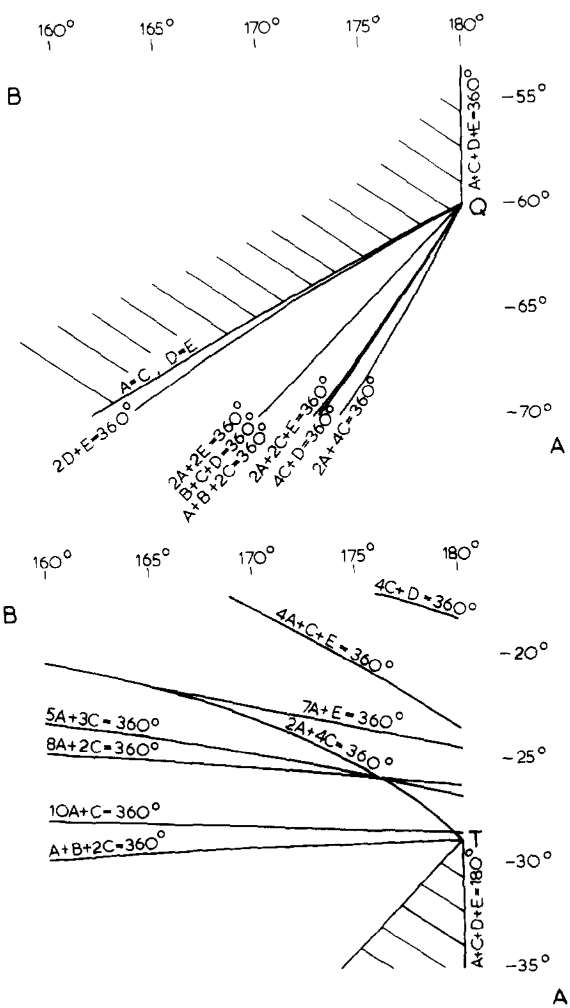


FIG. 8. Shows that certain angle relations are not satisfied by any equilateral convex pentagon.

The following 107 relations may be eliminated by use of the partial order $<$.

$<$ 01110
00111
01200
00201
11010
10011
11100
10101
21000
20001

$<$ 00021
00030

$>$ 20002
22000
21001
32000
31001
30002

$>$ 11200
01030
01120
02200
01201
00202
01210
01300
13000
11002
11020
12100

$>$ 10111
00031
02116
01111
00112
00121
00211
12001
10003
12010
11011
10012
10021
11101
10102
11110

$>$ 20201
00230
11120
10121
11210
10211
11300
10301
22010
21011
20012
21020
20021
22100
21101
20102
21110
20111
21200

$>$ 00410
00320
01400
00401

$>$ 40101
31020
30021
31110
30111
31200
30201
42000
41001
40002
41010
40011
41100
41110
40111
41200
40201
51010
50011
51100
50101

$>$ 20400
00600
10320
10410
10500
20040
20130
20220
21300
20301
20310
20500
30220
30310
30400
40400

$>$ 70001
61010
60011
61100
60101
71000
81000
80001

$>$ 50300
50120
50210
60300

This leaves 100 relations. Each of these is satisfied by pentagons lying on a curve crossing the region $\mathscr{P}$ (see Fig. 9).

LEMMA 4. *There are precisely* 100 *relations satisfied by some equilateral convex pentagon, namely*:

1. $A+2E=360°$
2. $A+B+E=360°$
3. $A+2B=360°$
4. $C+2E=360°$
5. $B+C+E=360°$
6. $2B+C=360°$
7. $B+2D=360°$
8. $D+2E=360°$
9. $B+D+E=360°$
10. $2B+D=360°$
11. $3E=360°$
12. $B+2E=360°$
13. $2B+E=360°$
14. $3B=360°$
15. $4A=360°$
16. $3A+C=360°$
17. $3A+D=360°$
18. $3A+E=360°$
19. $3A+B=360°$
20. $2A+2C=360°$
21. $2A+C+D=360°$
22. $2A+C+E=360°$
23. $2A+B+C=360°$
24. $2A+2D=360°$
25. $2A+D+E=360°$
26. $2A+B+D=360°$
27. $A+3C=360°$
28. $A+2C+D=360°$
29. $A+2C+E=360°$
30. $A+C+2D=360°$
31. $A+3D=360°$
32. $4C=360°$
33. $3C+D=360°$
34. $3C+E=360°$
35. $2C+2D=360°$
36. $C+3D=360°$
37. $4D=360°$
38. $5A=360°$
39. $4A+C=360°$
40. $4A+D=360°$
41. $4A+E=360°$
42. $4A+B=360°$
43. $3A+2C=360°$
44. $3A+C+D=360°$
45. $3A+C+E=360°$
46. $3A+B+C=360°$
47. $3A+2D=360°$
48. $3A+D+E=360°$
49. $3A+B+D=360°$
50. $2A+3C=360°$
51. $2A+2C+D=360°$
52. $2A+C+2D=360°$
53. $2A+3D=360°$
54. $A+4C=360°$
55. $A+3C+D=360°$
56. $A+2C+2D=360°$
57. $A+C+3D=360°$
58. $A+4D=360°$
59. $5C=360°$
60. $6A=360°$
61. $5A+C=360°$
62. $5A+D=360°$
63. $5A+E=360°$
64. $5A+B=360°$
65. $4A+2C=360°$
66. $4A+C+D=360°$
67. $4A+2D=360°$
68. $3A+3C=360°$
69. $3A+2C+D=360°$
70. $3A+C+2D=360°$
71. $3A+3D=360°$
72. $7A=360°$
73. $6A+C=360°$
74. $6A+D=360°$
75. $6A+E=360°$
76. $6A+B=360°$
77. $5A+2C=360°$
78. $5A+C+D=360°$
79. $5A+2D=360°$
80. $4A+3C=360°$
81. $4A+2C+D=360°$
82. $4A+C+2D=360°$
83. $4A+3D=360°$
84. $8A=360°$
85. $7A+C=360°$
86. $7A+D=360°$
87. $6A+2C=360°$
88. $6A+C+D=360°$
89. $6A+2D=360°$
90. $9A=360°$
91. $8A+C=360°$
92. $8A+D=360°$
93. $7A+2C=360°$
94. $7A+C+D=360°$
95. $7A+2D=360°$
96. $10A=360°$
97. $9A+C=360°$
98. $9A+D=360°$
99. $11A=360°$
100. $12A=360°$

4. PROOF OF MAIN THEOREM

We next observe that if an equilateral convex pentagon tiles the plane, it simultaneously satisfies at least *two* of the 100 relations. For, every angle of the tiling pentagon is involved in some such relation, yet no one relation involves all five angles. So we must consider intersections of the 100 relations (see Fig. 9). Six of the relations coincide.

LEMMA 5. *The following relations are equivalent*:

(A) 2. $A+B+E=360°$

(B) 23. $2A+B+C=360°$

(C) 25. $2A+D+E=360°$

(D) 35. $2C+2D=360°$

(E) 44. $3A+C+D=360°$

(F) 60. $6A=360°$.

Proof. If $A=60°$ then the pentagon is an equilateral triangle joined along one edge to a rhombus and it is clear that all 6 relations hold. Clearly $(A) \Leftrightarrow (D)$ since $A+B+C+D+E=540°$.

Now

$$A>60° \Rightarrow C+D>180° \Rightarrow 3A+C+D>360°$$

and

$$A<60° \Rightarrow C+D<180° \Rightarrow 3A+C+D<360°.$$

Hence $(A) \Leftrightarrow (D) \Leftrightarrow (E) \Leftrightarrow (F)$ and $(F) \Rightarrow (B)$, $(F) \Rightarrow (C)$.

We now show that $(B) \Rightarrow (F)$: Assume that $2A+B+C=360°$. Let $A=(0,0)$ and $B(1,0)$ then

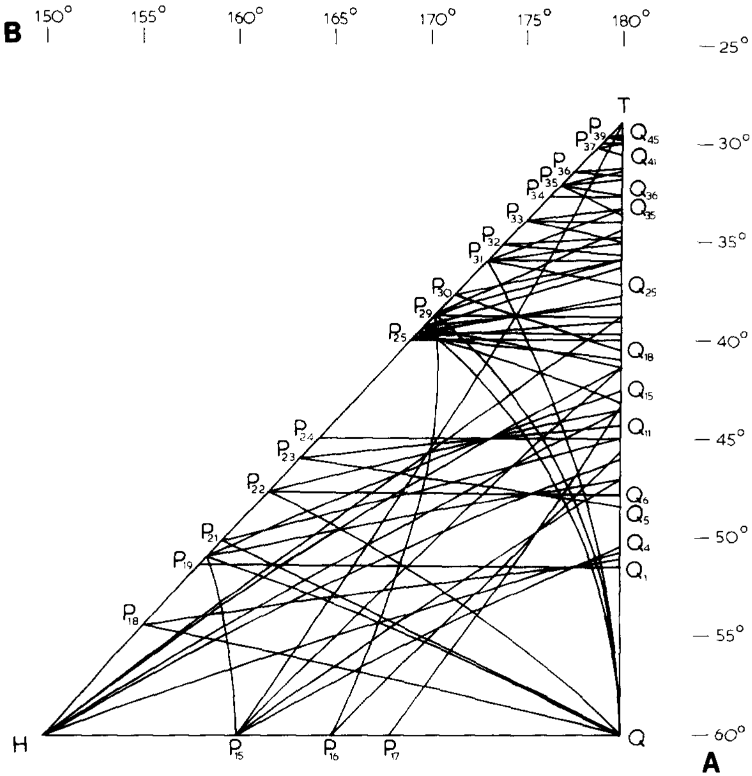
$$D=(\cos A, \sin A)$$
$$C=(1-\cos B, \sin B)$$
$$E=(1-\cos B-\cos(B+C-180°), \sin B+\sin(B+C-180°)).$$

Now $(DE)^2=1$. Hence

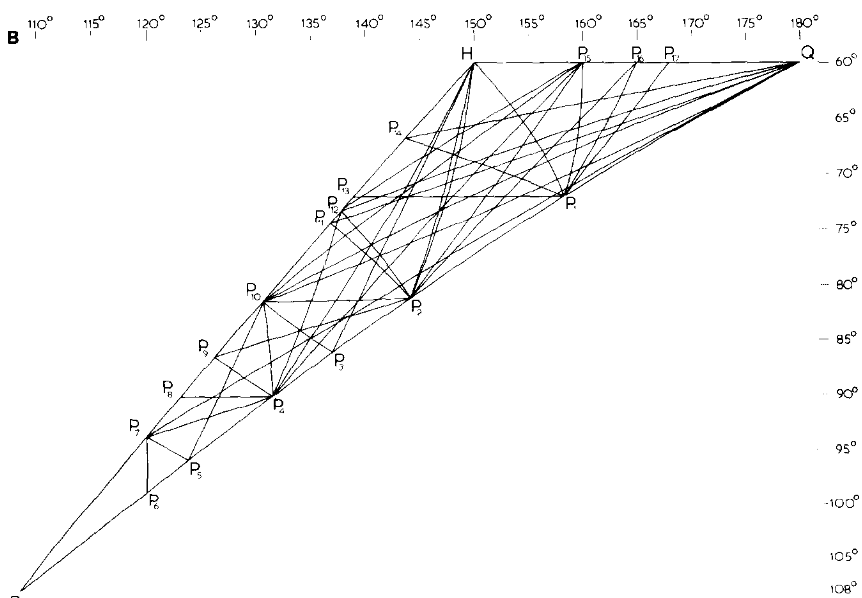
$$(\cos A-1+\cos B-\cos(B+C))^2+(\sin A-\sin B+\sin(B+C))^2=1.$$

Also $B+C=360°-2A$, so

$$\begin{aligned}1&+\cos^2 A+\cos^2 B+\cos^2 2A-2\cos A-2\cos B+2\cos 2A\\&+2\cos A\cos B-2\cos A\cos 2A-2\cos B\cos 2A+\sin^2 A\\&+\sin^2 B+\sin^2 2A-2\sin A\sin B-2\sin A\sin 2A\\&+2\sin B\sin 2A=1\end{aligned}$$



B
150°
155°
160°
165°
170°
175°
180°
25°
30°
35°
40°
45°
50°
55°
60°
T
H
Q
A



B
110°
115°
120°
125°
130°
135°
140°
145°
150°
155°
160°
165°
170°
175°
180°
60°
65°
70°
75°
80°
85°
90°
95°
100°
105°
108°
H
Q
R
A

or

$$4-2\cos A-2\cos B+2\cos 2A+2\cos(A+B)$$
$$-2\cos(2A-A)-2\cos(B+2A)=1$$

or

$$4-4\cos A+2\cos 2A=1+2\cos B+2\cos(2A+B)-2\cos(A+B)$$
$$=1+4\cos(A+B)\cos A-2\cos(A+B).$$

Hence

$$3-4\cos A+2\cos 2A=2\cos(A+B)(2\cos A-1)$$

or

$$1-4\cos A+4\cos^2 A=2\cos(A+B)(2\cos A-1).$$

Hence $\cos A=\frac{1}{2}$ or $\cos A-\cos(A+B)=\frac{1}{2}$, which is not possible in $\mathscr{P}$ except at R, and here $2A+B+C\neq 360°$. So $\cos A=\frac{1}{2}$ and $A=60°$. To show $(C)\Rightarrow(F)$ we argue as above with E replacing B. ▌

FIG. 9. Shows the 100 relations satisfied by some equilateral pentagon:

Curve	Endpoints		Curve	Endpoints		Curve	Endpoints		Curve	Endpoints	
1	H	Q_{22}	26	H	Q_{12}	51	Q	P_{20}	76	P_{37}	Q_{42}
2	Q	H	27	P_4	P_{12}	52	P_{20}	Q_7	77	P_{27}	Q_{14}
3	P_3	H	28	P_2	P_{12}	53	P_{20}	Q_{15}	78	P_{27}	Q_{20}
4	P_{10}	Q_7	29	P_2	T	54	P_1	P_{29}	79	P_{27}	Q_{24}
5	Q	P_{10}	30	Q	P_{12}	55	Q	P_{29}	80	P_{35}	Q_{33}
6	P_3	P_{10}	31	P_{12}	Q_4	56	P_{29}	Q_{22}	81	P_{35}	Q_{36}
7	P_4	Q_{16}	32	P_4	H	57	P_{29}	Q_{27}	82	P_{35}	Q_{37}
8	Q	P_{10}	33	P_2	H	58	P_{29}	Q_{31}	83	P_{35}	Q_{39}
9	P_4	P_{10}	34	P_2	Q_9	59	P_1	Q_{13}	84	P_{24}	Q_{10}
10	P_5	P_{10}	35	Q	H	60	Q	H	85	P_{26}	Q_{17}
11	Q	P_7	36	H	Q_8	61	Q	P_{18}	86	P_{26}	Q_{21}
12	P_4	P_7	37	H	Q_{16}	62	P_{18}	Q_2	87	P_{33}	Q_{29}
13	P_5	P_7	38	P_1	P_{13}	63	P_{30}	Q_{18}	88	P_{33}	Q_{32}
14	P_6	P_7	39	P_1	P_{14}	64	P_{30}	Q_{27}	89	P_{33}	Q_{34}
15	P_4	P_8	40	Q	P_{14}	65	Q	P_{22}	90	P_{25}	Q_{19}
16	P_4	P_9	41	Q	P_{21}	66	P_{22}	Q_6	91	P_{32}	Q_{28}
17	P_2	P_9	42	P_{21}	Q_{10}	67	P_{22}	Q_{12}	92	P_{32}	Q_{30}
18	P_2	P_{11}	43	P_1	H	68	Q	P_{28}	93	P_{39}	Q_{43}
19	Q	P_{11}	44	Q	H	69	P_{28}	Q_{16}	94	P_{39}	Q_{44}
20	P_4	P_{10}	45	Q	P_{31}	70	P_{28}	Q_{23}	95	P_{39}	Q_{45}
21	P_2	P_{10}	46	P_{31}	Q_{29}	71	P_{28}	Q_{26}	96	P_{31}	Q_{27}
22	P_2	H	47	H	Q_3	72	P_{19}	Q_1	97	P_{36}	Q_{38}
23	Q	H	48	P_{31}	Q_{25}	73	P_{23}	Q_5	98	P_{36}	Q_{40}
24	Q	P_{10}	49	P_{31}	Q_{34}	74	P_{23}	Q_{11}	99	P_{34}	Q_{35}
25	Q	H	50	P_1	P_{20}	75	P_{37}	Q_{41}	100	P_{38}	Q_{42}

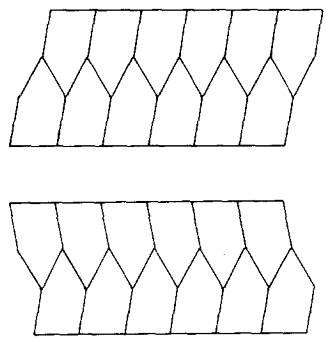


FIG. 10. Type 1 equilateral convex pentagons tile in a variety of ways.

Any equilateral convex pentagon which satisfies these 6 relations is said to be of Type 1.

The two relations

5. $B + C + E = 360°$

24. $2A + 2D = 360°$

coincide, and any equilateral convex pentagon which satisfies these is said to be of Type 2(a).

The two relations

9. $B + D + E = 360°$

20. $2A + 2C = 360°$

coincide, and any equilateral convex pentagon which satisfies these is said to be of Type 2(b). Together, Types 2(a) and 2(b) constitute Type 2.

Type 1 consists of all equilateral convex pentagons with two adjacent angles adding to 180°, while Type 2 consists of all equilateral convex pentagons with two nonadjacent angles adding to 180°. All these pentagons tile the plane (see Figs. 10, 11).

To find all other equilateral convex pentagons which tile the plane, we must consider all those which simultaneously satisfy at least two of the remaining 90 relations.

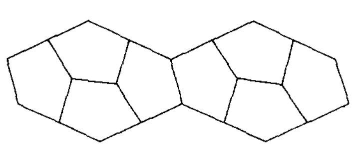


FIG. 11. Type 2 equilateral convex pentagons tile by forming hexagons with opposite pairs of sides parallel, which tile in strips.

We can shorten this task as follows: Any tiling pentagon satisfies at least one of the 14 relations involving B, namely:

3. $A+2B=360^\circ$
6. $2B+C=360^\circ$
7. $B+2D=360^\circ$
10. $2B+D=360^\circ$
12. $B+2E=360^\circ$
13. $2B+E=360^\circ$
14. $3B=360^\circ$
19. $3A+B=360^\circ$
26. $2A+B+D=360^\circ$
42. $4A+B=360^\circ$
46. $3A+B+C=360^\circ$
49. $3A+B+D=360^\circ$
64. $5A+B=360^\circ$
76. $6A+B=360^\circ$

Further, the relations involving B that it satisfies cannot all belong to the second half of this list, since in each of these $m_A > m_B$, and so, in the tiling there would be more angles A than B, an impossibility. Thus the pentagon satisfies at least two relations, including at least one of the seven relations in the first half of the above list. There are 54 such pentagons, satisfying the following sets of relations:

3, 6	7, 11	7, 62	10, 13
3, 17	7, 17	7, 54	10, 12
3, 11	7, 18	7, 65	10, 15
3, 7, 21	7, 28	7, 72	10, 11
3, 18	7, 8	7, 36	10, 16
3, 8, 28	7, 33	7, 52	10, 17
3, 19, 38	7, 38	7, 66	11, 12, 13, 14
3, 4, 30	7, 19	7, 73	
3, 31	7, 39	7, 26, 42	
3, 39	7, 30	7, 68	
3, 40	7, 43	7, 55	
	7, 40	7, 84	
6, 7	7, 61	7, 74	
6, 11, 32	7, 47	7, 45, 67	
6, 17	7, 51	7, 53	
6, 27	7, 41	7, 77	

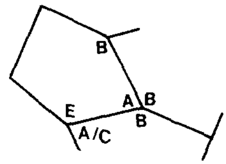


FIG. 12. Shows that a certain equilateral convex pentagon does not tile.

Of these 54, only three satisfy sets of relations involving all 5 angles:

3, 4, 30 $A+2B=360°$, $C+2E=360°$, $A+C+2D=360°$

3, 8, 28 $A+2B=360°$, $D+2E=360°$, $A+2C+D=360°$

7, 45, 67 $B+2D=360°$, $3A+C+E=360°$, $4A+2D=360°$

Of these, the third does not tile since $m'_D > m_B$ in the only relation involving B. The second does not tile; for suppose it does, and consider any tiling of it near a vertex where $A+2B=360°$. We see from Fig. 12 that E and A or E and C are forced together, but these combinations do not occur in any of the relations satisfied by that pentagon, yielding a contradiction.

This leaves only the first, which does tile the plane (see Fig. 13), and the theorem is proved.

5. FURTHER COMMENTS

A natural question to ask is: Given a tile which tiles the plane, can a description be given of all the distinct tilings of the plane with that tile?

For a general Type 1 or Type 2 tile (i.e., a tile which does not satisfy any additional relation) it is easy to answer this question. However for some tiles, and in particular the "versatile" $A=60°$, $B=160°$, $C=80°$, $D=100°$, $E=140°$, the question is hard to answer, since this tile satisfies no fewer than 11 angle relations. This tile and also the Type 2 tile with $A=72°$ yield tilings possessing only rotational symmetries illustrated in Figs. 14–16. One of these has appeared previously [1, p 159].

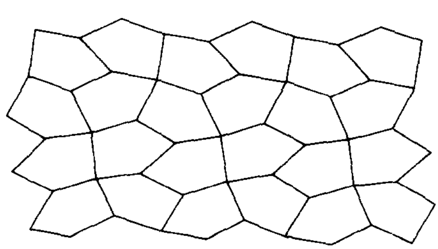


FIG. 13. The equilateral convex pentagon X with angles A, B, C, D, E satisfying $A+2B=360°$, $C+2E=360°$, $A+C+2D=360°$ tiles the plane.

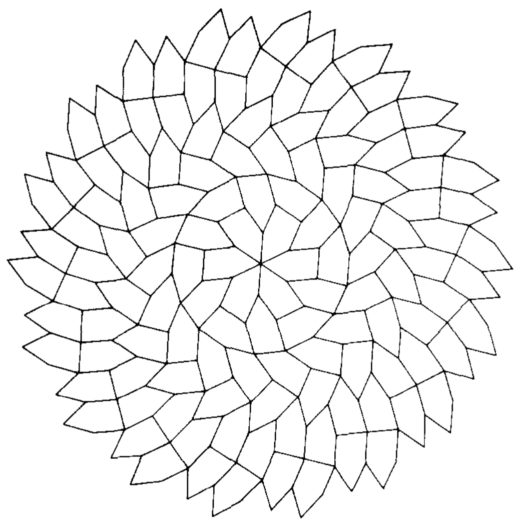


FIGURE 14

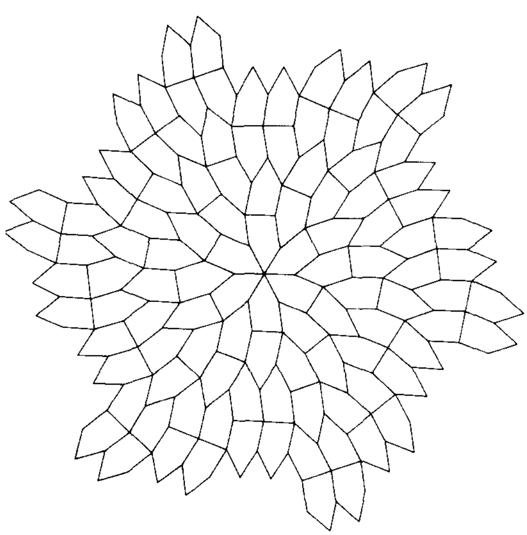


FIGURE 15

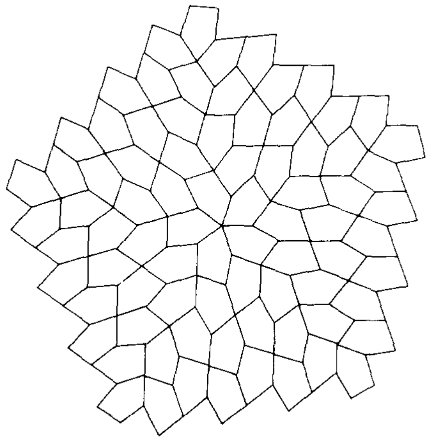


FIGURE 16

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2. D. SCHATTSCHNEIDER, Tiling the plane with congruent pentagons, *Math. Mag.* **51** (1978), 29–44.